JOURNAL OF APPROXIMATION THEORY **93**, 100–113 (1998)
ARTICLE NO. AT963090

Uniform Bounds for Sampling Expansions

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Communicated by Rolf J. Nessel

Received November 7, 1995; accepted in revised form October 15, 1996

Let $f \in B^2_\sigma$, i.e., $f \in L^2(\mathbb{R})$ and its Fourier transform $F(s) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i s t}\, dt$ vanishes outside of $[-\sigma, \sigma]$, then the Shannon sampling theorem says that f can be reconstructed by its infinitely many sampling points at $\{k/(2\sigma)\}$, $k \in \mathbb{Z}$, i.e.,

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2\sigma}\right) \frac{\sin \pi(2\sigma t - k)}{\pi(2\sigma t - k)}, \qquad \forall t \in \mathbb{R}.$$

But, in practice, only finitely many samples are available, so one would like to study the truncation error

$$T_N(t) = f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k), \qquad \forall f \in B^2_\sigma.$$

The error bounds commonly seen in literature are not uniform. In this paper, the author gives uniform bounds for the truncation error for $f \in B^2_\sigma$, when its Fourier transform satisfies some smooth conditions.

1. INTRODUCTION

The functions $f(x) \in L^2(\mathbb{R})$ whose Fourier transforms defined by

$$F(s) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i s t}\, dt$$

vanish outside of finite interval $[-\sigma, \sigma]$ are called bandlimited. The σ is the bandwidth. It is well known that the functions can be represented by the so called Shannon's expansion

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k), \tag{1.1}$$

0021-9045/98 $25.00

where

$$\operatorname{sinc}(t) := \int_{-1/2}^{1/2} e^{2\pi i s t}\, ds = \begin{cases} 1 & \text{if} \quad t = 0 \\ (\sin \pi t)/\pi t & \text{if} \quad t \neq 0 \end{cases}. \tag{1.2}$$

An important underlying engineering principle here is that all the information contained in such a bandlimited signal can be recovered from its equidistant samples. The sampling rate 2σ is known as the Nyquist rate.

Shannon's expansion requires us to know the exact values of f at infinitely many points and to sum an infinite series. In practice, only finitely many samples are available so we would like to develop bounds on the size of the truncation error

$$T_N(t) := f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k) \tag{1.3}$$

associated with (1.1). The error bounds commonly seen in the literature, e.g., [1, 3, 6, 8, 9, 12, 13], are not uniform. In this paper we will establish uniform bounds for $T_N(t)$ as given in (1.3).

The sinc(t) function (1.2) that appears in (1.1), (1.3) oscillates and decays like $1/t$ at $\pm\infty$. Our error estimates make use of a number of specialized bounds for sums of samples of sinc functions as well as other functions having similar properties. For organizational purposes we present all of these technical results in Section 2.

In Section 3, we describe previously published bounds on $T_N(t)$ and then show that

$$|T_N(t)| \leqslant \frac{C_1}{N^{r+1/2}}$$

$$|T_N(t)| \leqslant \frac{C_2 \ln N}{N^{t+1}}$$

when

$$|f(t)| \leqslant \frac{A}{|t|^{r+1}}, \qquad t \neq 0.$$

Here C_1 and C_2 are constants that depend on f and the parameters $r > 0$, $A > 0$. We also show that

$$|T_N(t)| \leqslant \left(1 + \frac{\pi}{2}\right) \left\{ \left| f\left(\frac{-N}{2\sigma}\right) \right| + \left| f\left(\frac{N}{2\sigma}\right) \right| \right\}$$

in the case where f is monotone on $(-\infty, -N]$ and $[N, \infty)$. Using this bound we estimate the error that results when we approximate some $f \in L^1(\mathbb{R})$ by the sinc polynomial

$$\sum_{k=-N}^{N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k).$$

2. BOUNDS FOR SERIES OF SINC FUNCTIONS

In this section we establish bounds for various series of sinc functions. The proofs are technical. To simplify notation we set $2\sigma = 1$ and work with $\operatorname{sinc}(t-k)$, $k = 0, \pm 1, \pm 2, \cdots$.

LEMMA 2.1. *For* $\infty < t < \infty$,

$$\sum_{k=-\infty}^{\infty} \operatorname{sinc}(t-k) = 1, \tag{2.1}$$

$$\sum_{k=-\infty}^{\infty} \operatorname{sinc}^2(t-k) = 1. \tag{2.2}$$

Proof. Let $t \in \mathbb{R}$ be fixed. The Fourier series of $e^{2\pi i s t}$ as a function of s on $[-\frac{1}{2}, \frac{1}{2}]$ is given by

$$e^{2\pi i s t} = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i s k}, \tag{2.3}$$

where

$$\begin{aligned} c_k &= \int_{-1/2}^{1/2} e^{2\pi i s t} e^{-2\pi i s k}\, ds \\ &= \int_{-1/2}^{1/2} e^{2\pi i s (t-k)}\, ds \\ &= \operatorname{sinc}(t-k). \end{aligned}$$

We obtain (2.1) by setting $s = 0$ in (2.3). We obtain (2.2) from (2.3) by using Parseval's identity. ∎

LEMMA 2.2. *Let $N > M$ be integers and let $-\infty < t < \infty$. Then*

$$\left|\sum_{k=M}^{N} \operatorname{sinc}(t-k)\right| \leqslant \frac{\pi}{2} + 1.$$

Proof.

$$\sum_{k=M}^{N} \operatorname{sinc}(t-k) = \sum_{k=M}^{N} \int_{-1/2}^{1/2} e^{\pi is(t-k)}\, ds$$

$$= \int_{-1/2}^{1/2} \sum_{k=M}^{N} e^{-2\pi isk} \cdot e^{2\pi ist}\, ds$$

$$= \int_{-1/2}^{1/2} \frac{e^{-2\pi isM} - e^{-2\pi is(N+1)}}{1-e^{-2\pi is}} \cdot e^{2\pi ist}\, ds$$

$$= \int_{-1/2}^{1/2} \frac{e^{\pi is(2t-2M+1)} - e^{\pi is(2t-2N-1)}}{e^{\pi is} - e^{-\pi is}}\, ds$$

$$= \int_{-1/2}^{1/2} \frac{\sin \pi s(2t-2M+1) + \sin \pi s(-2t+2N+1)}{2\sin \pi s}\, ds$$

$$= I(\alpha) + I(\beta),$$

where $\alpha = 2t-2M+1$, $\beta = -2t+2N+1$, and

$$I(\alpha) := \frac{1}{2} \int_{-1/2}^{1/2} \frac{\sin(\alpha\pi s)}{\sin(\pi s)}\, ds = \int_{0}^{1/2} \frac{\sin(\alpha\pi s)}{\sin(\pi s)}\, ds$$

with a similar expression for $I(\beta)$.

Using the inequality

$$\frac{2}{\pi} t \leqslant \sin t \leqslant t \qquad \text{when} \qquad 0 \leqslant t \leqslant \frac{\pi}{2},$$

we see that then $0 \leqslant \alpha \leqslant 1$, we have

$$I(\alpha) \leqslant \int_{0}^{1/2} \frac{\sin \alpha\pi s}{\sin \pi s}\, ds \leqslant \int_{0}^{1/2} \frac{\alpha\pi s}{2s}\, ds \leqslant \frac{\pi}{4}.$$

When $\alpha > 1$, we compute

$$I(\alpha) = \int_{0}^{1/2} \frac{\sin(\alpha\pi s)}{\sin(\pi s)}\, ds = I_1(\alpha) + I_2(\alpha),$$

where

$$I_1(\alpha) := \int_{0}^{1/2\alpha} \frac{\sin(\alpha\pi s)}{\sin(\pi s)}\, ds \leqslant \int_{0}^{1/2\alpha} \frac{\alpha\pi s}{2s}\, ds = \frac{\alpha\pi}{2} \cdot \frac{1}{2\alpha} = \frac{\pi}{4}$$

is necessarily nonnegative, and

$$I_2(\alpha) := \int_{1/2\alpha}^{1/2} \frac{\sin(\alpha\pi s)}{\sin(\pi s)}\, ds.$$

To bound $I_2(\alpha)$ we first write

$$\begin{aligned}
I_2(\alpha) &= \int_{1/2\alpha}^{1/2} \frac{1}{\sin(\pi s)}\, d\left(-\frac{\cos \alpha\pi s}{\alpha\pi}\right) \\
&= \frac{1}{\sin(\pi s)}\left(-\frac{\cos \alpha\pi s}{\alpha\pi}\right)\Bigg|_{1/2\alpha}^{1/2} - \int_{1/2\alpha}^{1/2} \frac{\cos(\alpha\pi s)}{\alpha\pi} \cdot \frac{\pi\cos \pi s}{(\sin \pi s)^2}\, ds \\
&= -\frac{\cos \alpha\pi(1/2)}{\alpha\pi} - \int_{1/2\alpha}^{1/2} \frac{\cos(\alpha\pi s)}{\alpha} \cdot \frac{\cos \pi s}{(\sin \pi s)^2}\, ds
\end{aligned}$$

and thereby find

$$\begin{aligned}
|I_2(\alpha)| &\leqslant \frac{1}{\alpha\pi} + \int_{1/2\alpha}^{1/2} \frac{1}{\alpha} \cdot \frac{1}{(\sin \pi s)^2}\, ds \\
&\leqslant \frac{1}{\alpha\pi} + \int_{1/2\alpha}^{1/2} \frac{1}{\alpha} \cdot \frac{1}{(2s)^2}\, ds \\
&= \frac{1}{\alpha\pi} + \frac{1}{4\alpha} \cdot (2\alpha - 2) \\
&= \frac{1}{\alpha\pi} + \frac{1}{2} - \frac{1}{2\alpha} \leqslant \frac{1}{2}.
\end{aligned}$$

Upon combining these results we see that

$$|I(\alpha)| = |I(|\alpha|)| \leqslant \begin{cases} (\pi/4) & \text{when} \quad |\alpha| \leqslant 1 \\ (\pi/4) + (1/2) & \text{when} \quad |\alpha| \geqslant 1. \end{cases}$$

It follows that

$$\left| \sum_{k=M}^{N} \operatorname{sinc}(t-k) \right| \leqslant |I(\alpha)| + |I(\beta)| \leqslant \frac{\pi}{2} + 1. \quad \blacksquare$$

Note. When $t \neq \pm 1, \pm 2, \cdots$ the sums

$$\sum_{k=M}^{N} |\operatorname{sinc}(t-k)| = |\sin \pi t| \sum_{k=M}^{N} \frac{1}{\pi |t-k|}$$

are not bounded.

LEMMA 2.3. *If $r>1$, then*

$$\frac{1}{r-1}\cdot\frac{1}{(N+1)^{r-1}}\leqslant\sum_{k=N+1}^{\infty}\frac{1}{k^r}\leqslant\frac{1}{r-1}\cdot\frac{1}{N^{r-1}}. \tag{2.4}$$

LEMMA 2.4. *For $-\infty<t<\infty$*

$$\sum_{k=-\infty}^{\infty}|\operatorname{sinc}(t-k)|^q\leqslant\begin{cases}1 & \text{if } q\geqslant 2\\ 2+(2/\pi)+(2/\pi)\cdot(1/q-1) & \text{if } 1<q<2\end{cases}. \tag{2.5}$$

Proof. When $q\geqslant 2$ we use (2.2) and write

$$\sum_{k=-\infty}^{\infty}|\operatorname{sinc}(t-k)|^q\leqslant\sum_{k=-\infty}^{\infty}|\operatorname{sinc}(t-k)|^2=1.$$

When $1<q<2$ we use the periodicity of the sum and (2.4) to write

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}|\operatorname{sinc}(t-k)|^q\\
&\leqslant\sup_{0\leqslant t\leqslant 1}\sum_{k=-\infty}^{\infty}|\operatorname{sinc}(t-k)|^q\\
&\leqslant\sup_{0\leqslant t\leqslant 1}\left(|\operatorname{sinc}(t)|^q+|\operatorname{sinc}(t-1)|^q+\sum_{k\neq 0,1}\frac{|\sin\pi(t-k)|^q}{\pi^q|t-k|^q}\right)\\
&\leqslant 2+\sup_{0\leqslant t\leqslant 1}\sum_{k\neq 0,1}\frac{1}{\pi^q|t-k|^q}\\
&\leqslant 2+\frac{1}{\pi}\sup_{0\leqslant t\leqslant 1}\left(\sum_{k=2}^{\infty}\frac{1}{(k-t)^q}+\sum_{k=-\infty}^{-1}\frac{1}{(t-k)^q}\right)\\
&\leqslant 2+\frac{1}{\pi}\left(2+2\sum_{k=2}^{\infty}\frac{1}{k^q}\right)\\
&\leqslant 2+\frac{2}{\pi}+\frac{2}{\pi}\cdot\frac{1}{q-1}. \quad\blacksquare
\end{aligned}$$

The following result can be used to provide us with examples of bandlimited functions.

THEOREM 2.5 (C. Eoff. [7]). *Let* ..., $c_{-2}, c_{-1}, c_0, c_1, c_2$, ... *be a sequence of complex constants and let* $\sigma > 0$. *If* $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$, *then the series*

$$\sum_{k=-\infty}^{\infty} c_k \operatorname{sinc}(2\sigma t - k)$$

converges uniformly and absolutely on $\mathbb{R}$ *to a function in* B_σ^2.

3. UNIFORM BOUNDS FOR THE TRUNCATION ERROR

We develop various uniform bounds on the symmetric truncation error

$$T_N(t) := f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k).$$

The error bounds commonly seen in the literature (e.g., [1, 3, 6, 8, 9, 12, 13]) are not uniform. Three such results are given below.

THEOREM 3.1. (K. Yao and J. B. Thomas [12]). *Let* $0 < \lambda < 1$, *let* $f \in B_{\lambda\sigma}^2$, *and let*

$$M := \sup_{t \in \mathbb{R}} |f(t)|.$$

Then

$$|T_N(t)| \leqslant \frac{M \sec(\lambda\pi/2)}{2\pi} \cdot |\sin 2\pi\sigma t| \cdot \left(\frac{1}{N + [2\sigma t + 1/2]} + \frac{1}{N - [2\sigma t + 1/2]} \right)$$

for $|2\sigma t| < N - \frac{1}{2}$.

This bound depends on t as well as the band reduction parameter λ with the factor $\sec(\lambda\pi/2)$ being unbounded as $\lambda \to 1-$.

THEOREM 3.2 (H. S. Piper, Jr. [9]). *Let* $0 < \lambda < 1$, *let* $f \in B_{\lambda\sigma}^2$, *and let*

$$E := \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

Then

$$|T_N(t)| \leqslant \frac{1}{\pi^{3/2}} \{E \tan(\lambda\pi/2)\}^{1/2} (1 + \sqrt{2}) \cdot |\sin 2\pi\sigma t| \left(\frac{1}{N + [t + (1/2)]} + \frac{1}{N - [t + (1/2)]} \right)$$

for $|t| < N - \frac{1}{2}$.

Again the bound depends on t and the band reduction parameter λ with the factor $\tan(\lambda\pi/2)$ being unbounded as $\lambda \to 1-$.

THEOREM 3.3 (P. L. Butzer, W. Engels, and U. Scheben [3]). *Let $\sigma > 0$, $r = 1, 2, \ldots$, and $F \in C^{r-1}(\mathbb{R})$ with $F(s) = 0$ for $|s| > \sigma$. Assume further that $F^{(r-1)}$ is absolutely continuous, that $F^{(r)}$ is of bounded variation, and that $F^{(r)}(s)$ is continuous at the points $s = \pm\sigma$. Then the truncation error associated with the σ-bandlimited function**

$$f(t) = \int_{-\sigma}^{\sigma} F(s)\, e^{2\pi i s t}\, ds$$

is bounded by

$$|T_N(t)| \leqslant \frac{2}{\pi} \cdot \left(\frac{\sigma}{\pi}\right)^{r+1} \cdot \frac{V_r}{r+1} \cdot |\sin 2\pi\sigma t| \cdot \frac{1}{(N - 2\sigma|t|)^{r+1}} \qquad \text{for} \quad |t| < \frac{N}{2\sigma}.$$

Here V_r is the variation of $F^{(r)}$ on $[-\sigma, \sigma]$.

An integration-by-parts argument in [3] shows that the hypotheses of Theorem 3.3 imply that

$$|f(t)| \leqslant \frac{A}{|t|^{r+1}} \qquad \text{when} \quad t \neq 0, \tag{3.1}$$

We will use (3.1) as the decay condition that we impose upon f to derive our uniform bounds. (A bound of the form

$$\left| f\left(\frac{k}{2\sigma}\right) \right| \leqslant \frac{C}{|k|^{r+1}}, \qquad k = \pm 1, \pm 2, \cdots$$

is all that is needed in our proof.)

THEOREM 3.4. *Let $f \in B_\sigma^2$ satisfy a decay condition of the form*

$$|(t)| \leqslant \frac{A}{|t|^{r+1}} \qquad t \neq 0$$

for some choice of $A > 0$ and $r > 0$, then

$$|T_N(t)| \leqslant A \cdot (2\sigma)^{r+1} \cdot \sqrt{\frac{2}{2r+1}} \cdot \frac{1}{N^{r+1/2}} \qquad \forall t \in \mathbb{R},\ N \in \mathbb{N}, \tag{3.2}$$

$$|T_N(t)| < A \cdot (2\sigma)^{r+1} \cdot \frac{2\sqrt{2e}}{\pi} \cdot \frac{(\pi + \ln N)}{N^{r+1}} \qquad \forall t \in \mathbb{R},\ N \geqslant 8. \tag{3.3}$$

Proof. Using Cauchy's inequality, (2.2), the decay hypothesis, and Lemma 2.3 in turn we write

$$\begin{aligned}
|T_N(t)| &= \left| \sum_{|k|>N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k) \right| \\
&\leqslant \left(\sum_{|k|>N} \left| f\left(\frac{k}{2\sigma}\right) \right|^2 \right)^{1/2} \cdot \left(\sum_{|k|>N} \operatorname{sinc}^2(2\sigma t - k) \right)^{1/2} \\
&\leqslant \left(\sum_{|k|>N} \left| f\left(\frac{k}{2\sigma}\right) \right| \right)^{1/2} \\
&\leqslant A \cdot (2\sigma)^{r+1} \left(2 \sum_{k=N+1}^{\infty} \frac{1}{k^{2r+2}} \right)^{1/2} \\
&\leqslant A \cdot (2\sigma)^{r+1} \left(\frac{2}{2r+1} \cdot \frac{1}{N^{2r+1}} \right)^{1/2} \\
&= A \cdot (2\sigma)^{r+1} \sqrt{\frac{2}{2r+1}} \cdot \frac{1}{N^{r+1/2}}.
\end{aligned}$$

This is (3.2). We next show this error bound can be improved from $O(1/(N^{r+(1/2)}))$ to $O(\ln N/N^{r+1})$.

We will use the decay hypothesis and Hölder inequality together with Lemmas 2.3, 2.4. Let $1 < q < 2$ and $2 < p < \infty$ be chosen so that $1/p + 1/q = 1$. Then

$$\begin{aligned}
|T_N(t)| &= \left| \sum_{|k|>N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k) \right| \\
&\leqslant \left(\sum_{|k|>N} \left| f\left(\frac{k}{2\sigma}\right) \right|^p \right)^{1/p} \cdot \left(\sum_{|k|>N} |\operatorname{sinc}(2\sigma t - k)|^q \right)^{1/q} \\
&\leqslant A \cdot (2\sigma)^{r+1} \cdot \left(2 \sum_{k=N+1} \frac{1}{k^{pr+p}} \right)^{1/p} \cdot \left(2 + \frac{2}{\pi} + \frac{2}{\pi} \frac{1}{q-1} \right)^{1/q} \\
&\leqslant A \cdot (2\sigma)^{r+1} \cdot \left(\frac{2}{pr+p-1} \right)^{1/p} \cdot \frac{1}{N^{r+(p-1)/p}} \cdot \left(2 + \frac{2}{\pi} + \frac{2}{\pi} \frac{1}{q-1} \right)^{1/q} \\
&= A \cdot (2\sigma)^{r+1} \cdot \left(\frac{2}{pr+p-1} \right)^{1/p} \cdot \frac{1}{N^{r+1/q}} \cdot \left(1 + \frac{2}{\pi} + \frac{2}{\pi} \frac{1}{q-1} \right)^{1/q}.
\end{aligned}$$

Now we set

$$p = \ln N, \qquad q = \frac{\ln N}{\ln N - 1}$$

(and assume $N > e^2 = 7.389...$ to ensure that $p > 2$). Then

$$\left(\frac{2}{pr+p-1}\right)^{1/p} < \sqrt{2},$$

$$N^{1/q} = N^{1-(1/\ln N)} = \frac{N}{e},$$

and

$$\left(2+\frac{2}{\pi}+\frac{2}{\pi}\,\frac{1}{q-1}\right)^{1/q} = \left(2+\frac{2}{\pi}\ln N\right)^{1-(1/\ln N)} \leqslant \frac{2}{\pi}\,(\pi+\ln N).$$

Upon combining these estimates we find

$$|T_N(t)| < A \cdot (2\sigma)^{r+1} \cdot \frac{2\sqrt{2}e}{\pi} \cdot \frac{(\pi+\ln N)}{N^{r+1}}. \quad \blacksquare$$

A slightly sharper error bound can obtained when the sampled data has monotone tails.

THEOREM 3.5. *Let ..., $c_{-2}, c_{-1}, c_0, c_1, c_2, ...$ be a sequence of nonnegative real numbers such that*

$$\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty,$$

and such that for some positive integer M,

$$c_M \geqslant c_{M+1} \geqslant c_{M+2} \geqslant \cdots$$

$$c_{-M} \geqslant c_{-M-1} \geqslant c_{-M-2} \geqslant \cdots.$$

Then

$$f(t) := \sum_{k=-\infty}^{\infty} c_k \operatorname{sinc}(2\sigma t - k)$$

defines a function in B^2_σ and the corresponding truncation error is bounded by

$$|T_N(t)| = \left|\sum_{|k|>N} c_k \operatorname{sinc}(2\sigma t - k)\right| \leqslant (c_{N+1} + c_{-N-1}) \cdot \left(\frac{\pi}{2}+1\right) \qquad \textit{for} \quad N \geqslant M.$$

Proof. The above bound is obtained by using Lemma 2.2 to show that the partial sums

$$\sum_{k=N}^{N'} \operatorname{sinc}(2\sigma t - k) \qquad \sum_{k=N}^{N'} \operatorname{sinc}(2\sigma t + k)$$

of the truncation error are uniformly bounded and then applying the following lemma of Abel. ∎

LEMMA 3.6 (Abel). *Let $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_N \geqslant 0$ and assume that the partial sums*

$$S_m := \sum_{k=1}^{m} b_k$$

of the numbers $b_1, ..., b_N$ are uniformly bounded, i.e., $|S_m| \leqslant A$ for $1 \leqslant m \leqslant N$. Then

$$\left| \sum_{k=1}^{N} a_k b_k \right| \leqslant a_1 A.$$

EXAMPLE. By Theorem 2.5, the series

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{1}{1+k^2} \operatorname{sinc}(2\sigma t - k),$$

$$f(t) = \sum_{k=-\infty}^{\infty} e^{-|k|} \operatorname{sinc}(2\sigma t - k),$$

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{1}{|k|+1} \operatorname{sinc}(2\sigma t - k)$$

define functions in B_σ^2 with the corresponding truncation error being bounded by

$$|T_N(t)| \leqslant \frac{2+r}{N^2+2N+2},$$

$$|T_N(t)| \leqslant (2+\pi)\, e^{-N},$$

$$|T_N(t)| \leqslant \frac{2+\pi}{N+2}.$$

A function f is said to be almost bandlimited if the Fourier transform F is not compactly supported but rather has small tails. Finding a bound for the error when an almost bandlimited function is approximated by the sampling expansion is a very practical issue, since the functions encountered in applications of the sampling theory are not always exactly bandlimited. An important bound was given by J. L. Brown, Jr. [2 or 13]. We will present his result and use it to estimate the error when the function is approximated with only finitely many samples.

THEOREM 3.7 (J. L. Brown, Jr. [2]). *Let the function $f\colon \mathbb{R} \to \mathbb{C}$ have the representation*

$$f(t) = \int_{\mathbb{R}} F(s)\, e^{2\pi i s t}\, ds,$$

where $F \in L^1(\mathbb{R})$, and let $\sigma > 0$ be given. Then the series

$$f_\sigma(t) := \lim_{N \to \infty} \sum_{k=-N}^{N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k)$$

converges pointwise on $\mathbb{R}$ and

$$|f(t) - f_\sigma(t)| \leqslant 2 \int_{|s| > \sigma} |F(s)|\, ds.$$

When $F \in L^1(\mathbb{R})$ is suitably regular, we can pick some $\sigma > 0$ to ensure that the aliasing error

$$A_\sigma := 2 \int_{|s| > \sigma} |F(s)|\, ds \tag{3.4}$$

is small. The complete sampling expansion

$$f_\sigma(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k)$$

will then provide a good approximation to $f(t)$. But in real world situations, only finitely many samples are available. For this reason we also need a bound on the truncation error

$$T_N(t) := \sum_{|k| > N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k). \tag{3.5}$$

In this case, however, the samples $\{f(k/2\sigma)\}$ may not have the nice properties (e.g., the square summability) possessed by samples of a bandlimited function. The Riemann–Lebesgue lemma, insures that if $F \in L^1(\mathbb{R})$, then $f(k/2\sigma) \to 0$, as $k \to \pm\infty$; the rate of this convergence, however, is unknown.

When only finitely many samples are used to approximate f, it is reasonable to assume that the remaining samples are small in some collective sense. We will use Theorem 3.7, (3.4) and (3.5), to write

$$\left| f(t) - \sum_{k=-N}^{N} f\left(\frac{k}{2\sigma}\right) \operatorname{sinc}(2\sigma t - k) \right| \leqslant |T_N(t)| + A_\sigma$$

and use arguments analogous to these given above to provide bounds for (3.5) in three situations.

(i) If $\sum_{k=-\infty}^{\infty} |f(k/2\sigma)|$ converges, then

$$|T_N(t)| \leqslant \sum_{|k|>N} \left| f\left(\frac{k}{2\sigma}\right) \right|.$$

(ii) If $\sum_{k=-\infty}^{\infty} |f(k/2\sigma)|^2$ converges, then

$$|T_N(t)| \leqslant \left(\sum_{|k|>N} \left| f\left(\frac{k}{2\sigma}\right) \right|^2 \right)^{1/2}.$$

(iii) If $\{f(k/2\sigma)\}$ is monotone for $k > M/2\sigma$ and for $k < -M/2\sigma$, then

$$|T_N(t)| \leqslant \left(\left| f\left(\frac{N}{2\sigma}\right) \right| + \left| f\left(\frac{-N}{2\sigma}\right) \right| \right) \left(\frac{\pi}{2} + 1 \right) \qquad \text{for} \quad N > M.$$

EXAMPLE. If

$$f(t) = e^{\pi t^2}, \qquad F(s) = e^{-\pi s^2},$$

then

$$|A_\sigma| = 2 \int_{|s|>\sigma} e^{-\pi t^2}\, dt < 4 \int_\sigma^\infty e^{-\pi\sigma t}\, dt = \frac{4}{\pi\sigma} e^{-\pi\sigma^2} \qquad \text{for} \quad \sigma > 0,$$

and (since the tails are monotone)

$$|T_N(t)| \leqslant (\pi + 2)\, e^{-\pi N^2} \qquad \text{for} \quad N = 1, 2....$$

EXAMPLE. If

$$f(t) = \frac{1}{1+t^2}, \qquad F(s) = \pi e^{-2\pi|s|},$$

then

$$|A_\sigma(t)| = 2\int_{|s|>\sigma} \pi e^{-2\pi|t|}\,dt < 4\pi\int_\sigma^\infty e^{-2\pi t}\,dt = 2e^{-\pi\sigma} \qquad \text{for} \quad \sigma > 0,$$

and (since the tails are monotone)

$$|T_N(t)| \leqslant \frac{2+\pi}{N^2+1} \qquad \text{for} \quad N = 1, 2, \ldots . \quad \blacksquare$$

ACKNOWLEDGMENTS

I thank Dr. David W. Kammler for leading me to this subject and teaching me so much, and I thank the referees for many valuable comments and suggestions which helped to improve the drafts.

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